Master-slave scheme and controlling chaos in the Braiman-Goldhirsch method

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This Brief Report presents a master-slave scheme to demonstrate explicitly how control chaos works in the Braiman-Goldhirsch method for the one-dimensional map system. The scheme also naturally explains why the anomalous responses arise in a periodically perturbed, unimodal map system. The extension of the master-slave scheme to the *D*-dimensional map is also presented. [S1063-651X(99)03404-2]

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Braiman and Goldhirsch (BG) [1] proposed a simple nonfeedback method, in contrast to the Ott-Grebogi-Yorke feedback scheme [2], to create stable periodic orbits from a chaos using weak periodic perturbations. Though there are some successful numerical and experimental demonstrations of the BG method [3–5], the periodicity and the stability condition of the resulting stable state was not identified analytically until recently [6]. If one considers a generic one-dimensional chaotic map under the influence of a periodic perturbation,

$$z_{n+1}=f(z_n)-y_n, \tag{1}$$

where y_n is the added weak perturbation with periodicity p. The resulting stable states to the period-p perturbation in a chaotic map system can only have the periodicity $q=kp$, where k is an integer number. Furthermore, using the linear stability analysis, one can deduce that the stability condition, for the output with the periodicity $q=kp$, is

$$|M|=\left|\prod_{j=1}^{k}\left[\prod_{\ell=1}^{p}\left(\frac{\partial f}{\partial z}\bigg|_{\bar{z}_{j\ell}}\right)\right]\right|<1. \tag{2}$$

Here, $\overline{z_{j\ell}}$ is the $j\ell$ times mapping of $\overline{z_1}$, and $\overline{z_1}$ are the roots of the periodicity condition

$$z=f(f(\cdots(f(f(z)-y_1)-y_2)-\cdots)-y_{kp-1})-y_{kp}. \tag{3}$$

Even though the analysis is presented in an elegant mathematical form in Ref. [6], providing a more intuitive picture to illustrate how a chaotic system can be controlled by weak periodic perturbations is still a worthwhile effort. In this Brief Report, we will introduce a conceptual picture, called the master-slave scheme, to explain how controlling chaos works in the BG method. This picture will also gives us a new handle to understand why the anomalous responses occur in a dynamical system under the influence of periodic perturbations. For example, when a period-2 perturbation with elements $\{y_1=a,\ y_2=0.2\}$ is added to a chaotic logistic map, the system becomes $z_{n+1}=4z_n(1-z_n)-y_n$. The antimonotonicity, concurrent creation and destruction of periodic orbits [7,8], appears in the bifurcation diagram for some values of a, see Fig. 1. It seems to be in opposition to the well-known numerical fact [9] that *the antimonotonicity could never appear in an unperturbed one-dimensional unimodal map system*. Moreover, referring to Fig. 1(a) and Fig. 1(b), one finds that, for perturbation strength a between

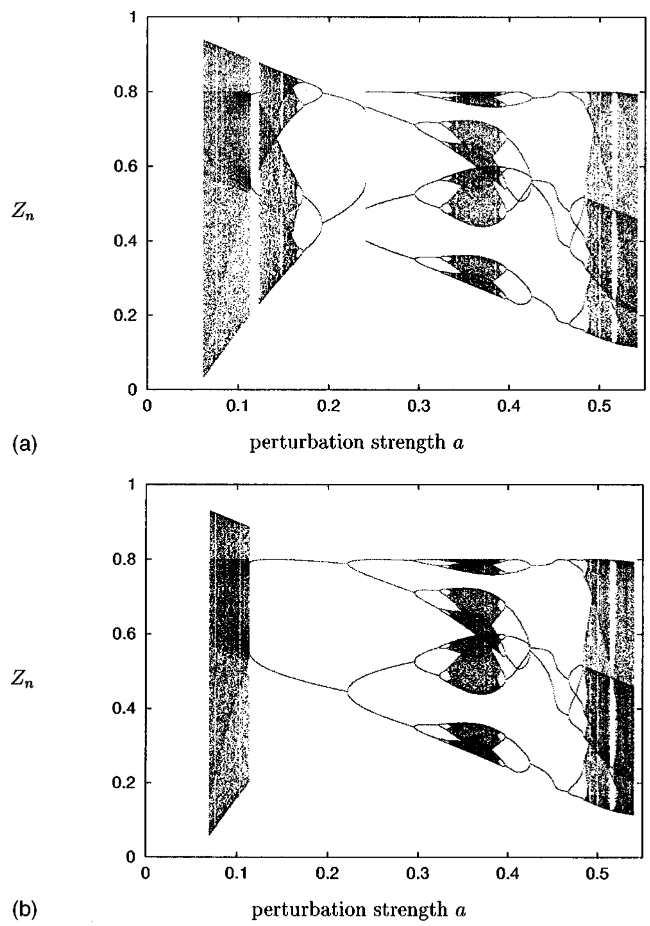

FIG. 1. Bifurcation diagrams for a periodically perturbed chaotic logistic map, $z_{n+1}=4z_n(1-z_n)-y_n$, where y_n is of period 2 and with elements $\{y_1=a,\ y_2=0.2\}$. With the initial values (a) $z_1=0.54$ and (b) $z_1=0.75$, 100 data points are collected and plotted after 4000 transient iterations.

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0.124 and 0.239, the system has two different attractors. This means that the system can become bistable when some suitable perturbations are included. This result also seems to be at odds with the fact that *the bistability could not occur in an unperturbed one-dimensional unimodal map*. In the next paragraph, we will demonstrate that these anomalous responses can be interpreted naturally in the master-slave scheme.

To begin with, let us consider a generic map under the influence of a period-p perturbation $\{y_1,y_2,\ldots,y_p\}$; see Eq. (1). For convenience, we will label the initial data and the initial perturbation as z_1 and y_1, respectively. The key idea of the master-slave scheme follows. We divide the original dynamical variables z_n into p new variables, called $\{x_m^{(1)},x_m^{(2)},\ldots,x_m^{(p)}\}$. The relation between z_n and the new variables $x_m^{(i)}$ is defined as

$$x_m^{(i)}=z_{pm+i}, \quad 1\leqslant i\leqslant p. \tag{4}$$

Hence, the original dynamical equation can be separated into p maps:

$$\begin{aligned}
x_m^{(2)}&=f(x_m^{(1)})-y_1,\\
x_m^{(3)}&=f(x_m^{(2)})-y_2,\\
&\cdots\\
x_m^{(p)}&=f(x_m^{(p-1)})-y_{p-1},\\
x_{m+1}^{(1)}&=f(x_m^{(p)})-y_p.
\end{aligned} \tag{5}$$

Plugging the first $(p-1)$ maps, $x_m^{(2)},x_m^{(3)},\ldots,x_m^{(p)}$, into $x_{m+1}^{(1)}$, one finds the map between $x_{m+1}^{(1)}$ and $x_m^{(1)}$, which characterizes the dynamical properties of the original system; let us call it *the master equation*:

$$\begin{aligned}
x_{m+1}^{(1)}&=F(x_m^{(1)};y_1,y_2,\ldots,y_p)\\
&=f(f(\cdots(f(f(x_m^{(1)})-y_1)-y_2)-\cdots)-y_{p-1})-y_p.
\end{aligned} \tag{6}$$

The remaining $p-1$ maps, which are just mappings of $x_m^{(1)}$, are designated *the slave equations*:

$$\begin{aligned}
x_m^{(2)}&=f(x_m^{(1)})-y_1,\\
x_m^{(3)}&=f(x_m^{(2)})-y_2,\\
&\cdots\\
x_m^{(p)}&=f(x_m^{(p-1)})-y_{p-1}.
\end{aligned} \tag{7}$$

Since the dynamics of the slave equations are completely controlled by the master equation, we name it *the master-slave scheme*. As long as the master equation, Eq. (6), is in a stable period-k orbit, then the slave equations indicate that $(p-1)$ images will appear simultaneously. This means that there exists a period-kp orbit in the original system.

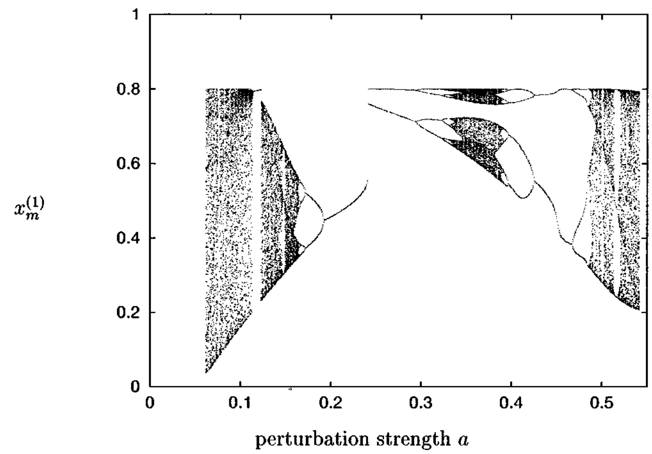

FIG. 2. Bifurcation diagram for the master equation, Eq. (10), versus the perturbation a. The initial point for $x_0^{(1)}$ is 0.54, and 50 data points are plotted after 2000 transient iterations.

From linear stability analysis, one can deduce that the stability condition for the period-k orbit in the master equation is $|M|<1$. The stability quantity M now simply is

$$M=\left(\frac{\partial}{\partial x}F^k(x_m^{(1)};y_1,y_2,\ldots,y_p)\right)\bigg|_{\overline{x^{(1)}}}, \tag{8}$$

where $\overline{x^{(1)}}$ is one of the roots of the periodicity condition

$$\overline{x^{(1)}}=F^k(\overline{x_m^{(1)}};y_1,y_2,\ldots,y_p). \tag{9}$$

Obviously, in terms of the original dynamical variable z_n and the map f, Eq. (8) and Eq. (9) will reduce to Eq. (2) and Eq. (3), respectively.

Now, to be more specific, let us take the perturbation y_n to be of period 2 with elements $\{y_1=a,\ y_2=0.2\}$. We will further assume that the system is a chaotic logistic map, $f(z)=4z(1-z)$, before we turn on the perturbation. In this special case, the master equation becomes

$$x_{m+1}^{(1)}=4[4x_m^{(1)}(1-x_m^{(1)})-a][1-4x_m^{(1)}(1-x_m^{(1)})+a]-0.2, \tag{10}$$

and the slave map is

$$x_m^{(2)}=4x_m^{(1)}(1-x_m^{(1)})-a. \tag{11}$$

Here, $x_m^{(1)}$ ($x_m^{(2)}$) denotes the odd (even) part of z_m, i.e., $x_m^{(1)}=z_{2m+1}(x_m^{(2)}=z_{2m+2})$, and the initial value is labeled $x_0^{(1)}=z_1$. The bifurcation diagram of the master equation, Eq. (10), with initial value $x_0^{(1)}=0.54$ and for the perturbation a with values between 0.0 and 0.55, is shown in Fig. 2. The bifurcation diagram indicates that the desired stable period-k orbit will appear if a suitable perturbation is applied on this chaotic logistic map. For example, period-1 orbit occurs when a is between (0.194,0.240); and period-2 orbits can be generated when a is at (0.170,0.194), (0.240,0.290), or (0.425,0.464); etc. From Eq. (11), the bifurcation diagram of the slave map is plotted in Fig. 3. One can see that the same periodic orbits also appear exactly in the same regions of the perturbation a. Obviously, the combination of Fig. 2

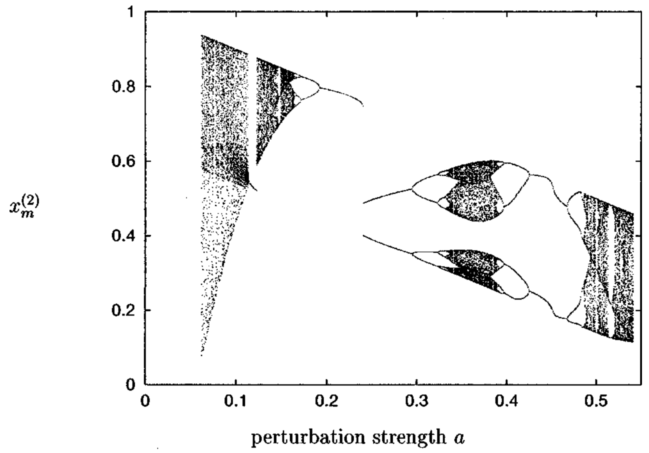

FIG. 3. The image $x_m^{(2)}$, which is determined by the slave equation (11), of the master equation (10) versus perurbation strength a.

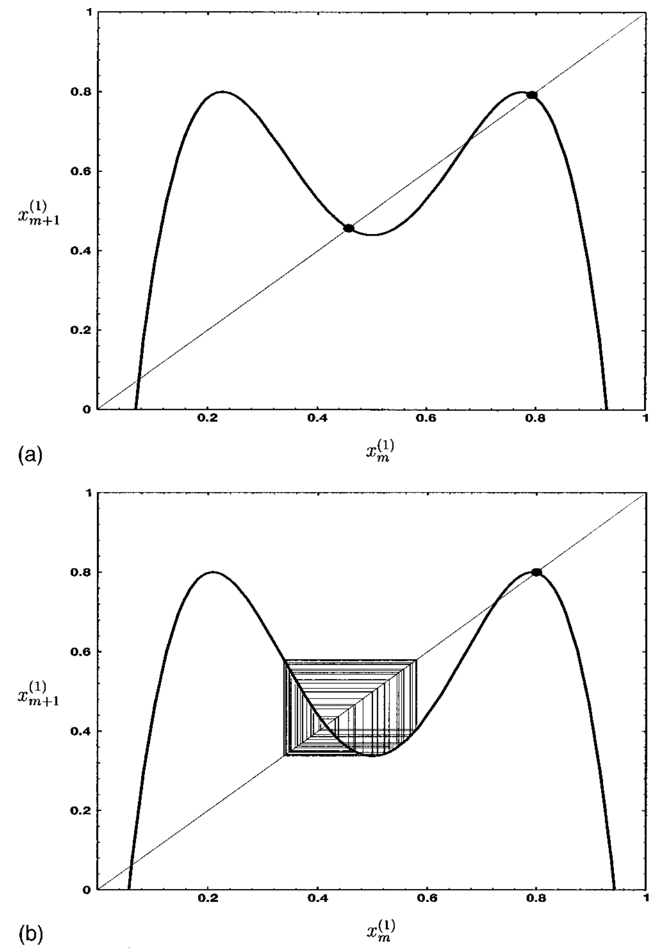

FIG. 4. (a) The map of the master equation, Eq. (10), with $a=0.2$, shows that it contains two fixed points at $\overline{x}=0.453$ and 0.793. (b) There are two attractors, one chaotic and the other one a fixed point located at $\overline{x}=0.80$, for the map of the master equation, Eq. (10), with $a=0.14$.

and Fig. 3 leads to Fig. 1(a) exactly. Also, from the numerical simulation presented in Fig. 1, one can clearly see that the periodicity of the stable response in the chaotic map under the influence of a period-2 perturbation is $2k$—as one would have expected.

A careful reader may have noticed that there are some anomalous responses in the bifurcation diagrams of this perturbed logistic map, which occurs at the region $a \in (0.25,0.45)$, see Figs. 1–3. These anomalous responses are called the antimonotonicity that does not exist in an unperturbed logistic map. Since the logistic map $f(z)=rx(1-x)$ only has one critical point at $x=0.5$, it seems to be a dilemma for those who are familiar with the work of Dawson, Grebogi, and Koçak [9]: *if a one-dimensional map* $x_{n+1}=F(x_n,\alpha)$ *has at least two critical points that lie in a chaotic attractor for a parameter* $\alpha=\alpha^*$, *then generally, F is antimonotone at* α^*. Futhermore, Fig. 1 indicates that the system could fall into different attractors for different initial values when suitable perturbation strength is applied. The occurrence of the bistability also cannot be explained in the framework of an unperturbed one-dimensional unimodal map. However, as has been mentioned in the preceding paragraph, the dynamics of a periodically perturbed system is governed by the master equation. The right hand side of Eq. (10) is a fourth order polynomial of $x_m^{(1)}$, and this implies that the system is multimodal. Therefore, the antimonotonicity and bistability could arise naturally in a periodically perturbed logistic map. For demonstration, we plot the maps of the master equation, Eq. (10), with $a=0.2$ and 0.14 in Fig. 4(a) and Fig. 4(b), respectively. One finds that there are two attractive fixed points in Fig. 4(a), and two attractors—one is chaotic and the other one is a fixed point in Fig. 4(b).

From the master-slave scheme, one knows that the master equation contains almost all the dynamical information and it also exhibits the complexity of the dynamics which results from the periodic perturbation, such that the antimonotonicity and the bistability can be induced in the perturbed unimodal map as shown in the preceding paragraph. Naturally it is expected that the periodic perturbation could also increase the complexity of the dynamics in the multimodal map. From numerical results, we find that the complexity of the bifurcation diagrams is indeed increased in the perturbed multimodal maps. However, owing to the complexity of the original and the perturbed unimodal maps, the identification of the anomalous responses in this type of system is unsuccessful.

The application of the master-slave scheme also can be easily extended to a D-dimensional map. Let us consider a generic D-dimensional map under the influence of a D-dimensional periodic perturbation,

$$\mathbf{z}_{n+1}=\mathbf{f}(\mathbf{z}_n)-\mathbf{y}_n . \qquad (12)$$

Here, $\mathbf{z}_n$ denotes the D-dimensional vector $\{z_n^1,\ldots,z_n^D\}$, $\mathbf{f}$ denotes the D-dimensional map $\{f^1,\ldots,f^D\}$, and the D-dimensional perturbations $\mathbf{y}_n$, $\{y_n^1,\ldots,y_n^D\}$, have periodicities $\{p^1,\ldots,p^D\}$ for each component. Based on the same arguments as in Ref. [6], one finds the periodicity of the output is kp when the periodic perturbations are applied with suitable strength. Here, p is the least common multiple of $\{p^1,\ldots,p^D\}$. The stability of the kp orbits is determined by the eigenvalues $\lambda_1,\lambda_2,\ldots,\lambda_D$ of the matrix $\mathbf{M}$ with components

$$\mathbf{M}_{ab} = \sum_{\ell_1, \ell_2, \ldots, \ell_d} \left(\frac{\partial \mathbf{f}^a}{\partial \mathbf{z}^{\ell_1}} \bigg|_{\overline{\mathbf{z}}_{kp}} \right) \times \left(\frac{\partial \mathbf{f}^{\ell_1}}{\partial \mathbf{z}^{\ell_2}} \bigg|_{\overline{\mathbf{z}}_{kp-1}} \right) \cdots \left(\frac{\partial \mathbf{f}^{\ell_{kp-2}}}{\partial \mathbf{z}^{\ell_{kp-1}}} \bigg|_{\overline{\mathbf{z}}_2} \right) \left(\frac{\partial \mathbf{f}^{\ell_{kp-1}}}{\partial \mathbf{z}^b} \bigg|_{\overline{\mathbf{z}}_1} \right). \tag{13}$$

Here, $\overline{\mathbf{z}}_j$ is the j times mapping of $\overline{\mathbf{z}}_1$, and $\overline{\mathbf{z}}_1$ are the roots of the periodicity conditions

$$\mathbf{z} = \mathbf{f}(\mathbf{f}(\cdots(\mathbf{f}(\mathbf{f}(\mathbf{z}) - \mathbf{y}_1) - \mathbf{y}_2) - \cdots) - \mathbf{y}_{kp-1}) - \mathbf{y}_{kp}. \tag{14}$$

Again, to apply the master-slave scheme to a D-dimensional map, we divide the original vector $\mathbf{z_n}$ into p new vectors, called $\{\mathbf{x}_m^{(1)}, \mathbf{x}_m^{(2)}, \ldots, \mathbf{x}_m^{(p)}\}$. The relation between $\mathbf{z}_n$ and the new vectors $\mathbf{x}_m^{(i)}$ are

$$\mathbf{x}_m^{(i)} = \mathbf{z}_{pm+i}, \quad 1 \leqslant i \leqslant p. \tag{15}$$

Then, the master equation is

$$\begin{aligned} \mathbf{x}_{m+1}^{(1)} &= \mathbf{F}(\mathbf{x}_m^{(1)}; \mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p) \\ &= \mathbf{f}(\mathbf{f}(\cdots(\mathbf{f}(\mathbf{f}(\mathbf{x}_m^{(1)}) - \mathbf{y}_1) - \mathbf{y}_2) - \cdots) - \mathbf{y}_{p-1}) - \mathbf{y}_p, \end{aligned} \tag{16}$$

and the $p-1$ slave equations are

$$\begin{aligned} \mathbf{x}_m^{(2)} &= \mathbf{f}(\mathbf{x}_m^{(1)}) - \mathbf{y}_1, \\ \mathbf{x}_m^{(3)} &= \mathbf{f}(\mathbf{x}_m^{(2)}) - \mathbf{y}_2, \\ &\cdots \\ \mathbf{x}_m^{(p)} &= \mathbf{f}(\mathbf{x}_m^{(p-1)}) - \mathbf{y}_{p-1}. \end{aligned} \tag{17}$$

Clearly, the master-slave scheme can also be extended to a D-dimensional mapping system. However, we also could not identify any anomalous responses in this type of system.

The master-slave scheme presented in this report gives us a conceptual picture that the periodic perturbation indeed makes controlling chaos feasible. In this scheme, the master equation contains almost all the dynamical information, in particular, it determines whether the attractor is chaotic or periodic. If one wants to know where in phase space the system is at other times, then the slave equation must be used. The scheme also helps us to understand explicitly how the anomalous responses, antimonotonicity and bistability, arise in a periodically perturbed one-dimensional unimodal map.

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